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Self-similar asymptotics of solutions to heat equation with inverse square potential

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Abstract. We show that the large-time behavior of solutions to the Cauchy problem for the linear heat equation with the inverse square potential is described by explicit self-similar solutions.

1. Introduction

In this paper, we study properties of weak solutions to the linear and singular initial value problem

$$u_t = \Delta u + \frac{\lambda}{|x|^2} u \quad \text{on} \quad \mathbb{R}^n \times (0, \infty) \tag{1.1}$$

$$u(x, 0) = u_0(x), \tag{1.2}$$

where $n \geqslant 3$, the parameter $\lambda \in \mathbb{R}$ is given and assumptions on the initial condition u_0 are stated below.

The initial value problem (1.1)–(1.2) was popularized by Baras and Goldstein [1] who discovered the "instantaneous blow up" of solutions, namely, the fact that Cauchy problem (1.1)–(1.2) has no positive local in time solutions if $\lambda > \frac{(n-2)^2}{4}$ (see also [7] for a simple proof via the Harnack inequality). Moreover, for $0 < \lambda \leqslant \frac{(n-2)^2}{4}$, the authors of [1] found necessary and sufficient conditions for u_0 so that a nonnegative solution exists. Note that the number $\frac{(n-2)^2}{4}$ appears as the best constant in the Hardy inequality

$$\int_{\mathbb{R}^n} |\nabla u(x)|^2 \, \mathrm{d}x \geqslant \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^2} \, \mathrm{d}x, \tag{1.3}$$

which is valid for $n \geqslant 3$ and for any $u \in H^1(\mathbb{R}^n)$. Below, in Sect. 3, we explain the role of the Hardy inequality (1.3) in the proof of existence of solutions to (1.1)–(1.2) for any $u_0 \in L^2(\mathbb{R}^n)$ and for $\lambda \in \left(0, \frac{(n-2)^2}{4}\right)$.

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This paper is motivated by the results of Vázquez and Zuazua [13] who studied the large-time behavior of solutions to (1.1)–(1.2) with $0 < \lambda \leqslant \frac{(n-2)^2}{4}$. Using a new weighted version of the Hardy–Poincaré inequality, they showed [13, Theorem 10.3] the convergence of some solutions toward radially symmetric, explicit self-similar solution. Here, we prove a result on the large-time behavior of solutions to problem (1.1)–(1.2) which is closely related to the one by Vázquez and Zuazua [13] and which is based on the estimates of the kernel of the Schrödinger operator $Hu = -\Delta u - \frac{\lambda}{|x|^2}u$, obtained recently by Liskevich and Sobol [8], Milman and Semenov [9], and Moschini and Tesei [10] (see Remark 2.4 for more details).

Let us mention that there is extensive literature on properties of the Schrödinger semigroup of linear operators e^{-tH_V} generated by $H_V \equiv -\Delta + V$, where a potential $V = V(x)$ is less singular at the origin, for example, when it belongs to the so-called Kato class (see [8,12] for additional references). This property is not shared by $V(x) = \frac{\lambda}{|x|^2}$ and it is well known that such singular potential belongs to a borderline case, where both the strong maximum principle and the Gaussian bound of the fundamental solution fail to hold.

The paper is organized as follows: In Sect. 2, we formulate our main result. In Sect. 3, we discuss the existence of solutions to problem (1.1)–(1.2). In Sect. 4, we recall properties of the fundamental solution of Eq. (1.1). Finally, in Sect. 5, we prove our main result stated in Theorem 2.1.

2. Main results

First, let us briefly review an asymptotic result for the initial value problem for the classical heat equation

$$\begin{aligned} u_t &= \Delta u \quad \text{on} \quad \mathbb{R}^n \times (0, \infty), \\ u(x, 0) &= u_0(x), \end{aligned} \tag{2.4}$$

which was the main motivation for our paper. If we assume that $u_0 \in L^1(\mathbb{R}^n)$, we obtain the following convergence

$$\lim_{t\to\infty} t^{\frac{n}{2}(1-\frac{1}{p})} \|u(\cdot, t) - MG(\cdot, t)\|_{L^p(\mathbb{R}^n)} = 0 \quad \text{for every} \quad p \in [1, \infty], \tag{2.5}$$

where $u = u(x, t)$ is the solution of problem (2.4) given as the convolution $u(t) = G(t) * u_0$, the quantity $M = \int_{\mathbb{R}^n} u_0(x)\, \mathrm{d}x = \int_{\mathbb{R}^n} u(x, t)\, \mathrm{d}x$ is constant in time, and $G(x, t) = \left(4\pi t\right)^{-n/2} \exp(-|x|^2/4t)$ is the heat kernel. To prove the relation in (2.5), it suffices to use the explicit formula for solutions to problem (2.4). In the first step, one should assume that u_0 is a smooth, compactly supported function and use the Taylor expansion of the heat kernel $G(x-y, t)$ (see, for example, [6, Theorem in Sect. 1.1.4]). Next, to complete the proof of (2.5) for any $u_0 \in L^1(\mathbb{R}^n)$, it suffices to approximate this initial datum by a sequence of smooth, compactly supported functions and to use the well-known inequality $\|G(t) * u_0\|_p \leqslant \|u_0\|_p$ for every $p \in [1, \infty]$. More information about the asymptotic expansion of solutions to the heat equation can be found in the paper by Duoandikoetxea and Zuazua [4].

The goal of this note is to show an analogous result for initial value problem (1.1)–(1.2) and, due to the nonexistence result by Baras and Goldstein, we have to assume that $\lambda < \frac{(n-2)^2}{4}$. By a technical obstacle, our method does not work in the critical case $\lambda = \frac{(n-2)^2}{4}$, see Remark 2.4, below.

The fundamental role in this paper is played by the parameter $\sigma = \sigma(\lambda)$ defined by the formula

$$\sigma = \frac{n-2}{2} - \sqrt{\frac{(n-2)^2}{4} - \lambda}, \tag{2.6}$$

which is a smaller root of a quadratic equation

$$\sigma^2 - (n-2)\sigma + \lambda = 0. \tag{2.7}$$

Moreover, the number $\sigma = \sigma(\lambda)$ satisfies

$$0 \leqslant \sigma(\lambda) < \frac{n-2}{2} \quad \text{for} \quad 0 \leqslant \lambda < \frac{(n-2)^2}{4} \quad \text{and} \quad \sigma(\lambda) < 0 \quad \text{for} \quad \lambda < 0. \tag{2.8}$$

To state our result, we recall a weight function, which is systematically used in the works [8–10]:

$$\varphi_\sigma(x,t) = \begin{cases} \left(\frac{\sqrt{t}}{|x|}\right)^\sigma & \text{if } \ |x| \leqslant \sqrt{t}, \\ 1 & \text{if } \ |x| \geqslant \sqrt{t}. \end{cases} \tag{2.9}$$

Moreover, we connect with this weight time-dependent norms defined as follows:

$$\|f\|_{q,\varphi_\sigma(t)} = \left(\int_{\mathbb{R}^n} |f(x)|^q \varphi_\sigma^{2-q}(x,t)\,\mathrm{d}x\right)^{\frac{1}{q}} \quad \text{for every} \quad 1 \leqslant q < \infty \tag{2.10}$$

$$\text{and} \quad \|f\|_{\infty,\varphi_\sigma(t)} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \varphi_\sigma^{-1}(x,t)|f(x)| \quad \text{for} \quad q = \infty. \tag{2.11}$$

In other words, $\|f\|_{q,\varphi_\sigma(t)} = \|\varphi_\sigma^{2/q-1}(\cdot,t) f\|_q$ for every $1 \leqslant q \leqslant \infty$, where $\|\cdot\|_q$ denotes the standard norm in the Lebesgue space $L^q(\mathbb{R}^n)$. Note that for $q = 2$, the norm $\|\cdot\|_{2,\varphi_\sigma(t)}$ agrees with the usual L^2-norm on $\mathbb{R}^n$.

Below, in Proposition 3.3, we show that for every $\lambda \in \left(-\infty, \frac{(n-2)^2}{4}\right)$ and each $u_0 \in L^2(\mathbb{R}^n)$, problem (1.1)–(1.2) has a unique global-in-time weak solution in a standard energy space. Moreover, the operator $-H = \Delta u + \frac{\lambda}{|x|^2}$ generates a semigroup of linear operators on $L^2(\mathbb{R}^n)$ (see Proposition 3.4 below) and the solution agrees with a semigroup solution. In addition, due to the results from [8,9] (see also [10,14,15]), this solution has the integral representation

$$u(x,t) = \int_{\mathbb{R}^n} K(x,y,t) u_0(y)\,\mathrm{d}y,$$

where the kernel K satisfies estimates recalled in Theorem 4.1 below. This integral formulation allows us to study solutions of problem (1.1)–(1.2) with initial conditions $u_0 \in L^1(\mathbb{R}^n)$ satisfying $|\cdot|^{-\sigma} u_0 \in L^1(\mathbb{R}^n)$.

Our main result on the large-time asymptotics of solutions to problem (1.1)–(1.2) is stated in the following theorem.

THEOREM 2.1. *Assume that $u = u(x,t)$ is the solution of problem (1.1)–(1.2) with $\lambda \in \left(-\infty, \frac{(n-2)^2}{4}\right)$ and with $u_0 \in L^1(\mathbb{R}^n)$ satisfying $|\cdot|^{-\sigma} u_0 \in L^1(\mathbb{R}^n)$, where σ is defined in (2.6). Then, for every $1 \leqslant q \leqslant \infty$*

$$\lim_{t\to\infty} t^{\frac{n}{2}(1-\frac{1}{q})-\frac{\sigma}{2}} \|u(\cdot,t) - AV(\cdot,t)\|_{q,\varphi_\sigma(t)} = 0, \tag{2.12}$$

where

$$A = \frac{\int_{\mathbb{R}^n} |x|^{-\sigma} u_0(x)\,\mathrm{d}x}{\int_{\mathbb{R}^n} |x|^{-2\sigma} \mathrm{e}^{-\frac{|x|^2}{4}}\,\mathrm{d}x} \tag{2.13}$$

and the function φ

$$V(x,t) = t^{\sigma-\frac{n}{2}} |x|^{-\sigma} \mathrm{e}^{-\frac{|x|^2}{4t}}$$

is the self-similar solution of (1.1).

Observe that by (2.6) for $\lambda = 0$, we have $\sigma = 0$ and the relation (2.12) reduces to (2.5) since the value of the denominator of A is equal to $(4\pi)^{n/2}$.

REMARK 2.2. *Note that, analogously as the constant $M = \int_{\mathbb{R}^n} u(x,t)\,\mathrm{d}x$ in the case of the heat Eq. (2.4), the following quantity $\widetilde{A} = \int_{\mathbb{R}^n} |x|^{-\sigma} u(x,t)\,\mathrm{d}x$ is also constant in time for sufficiently regular solutions to the initial value problem (1.1)–(1.2). Indeed, if we multiply Eq. (1.1) by the function $|x|^{-\sigma}$ and integrate over $\mathbb{R}^n$, we get, for a sufficiently regular solution that decays sufficiently fast as $|x| \to \infty$, the following equality*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^n} |x|^{-\sigma} u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^n} |x|^{-\sigma} \Delta u(x,t)\,\mathrm{d}x + \lambda \int_{\mathbb{R}^n} |x|^{-\sigma-2} u(x,t)\,\mathrm{d}x.$$

Now, integration by parts gives us

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^n} |x|^{-\sigma} u(x,t)\,\mathrm{d}x = \left(\sigma^2 - \sigma(n-2) + \lambda\right) \int_{\mathbb{R}^n} |x|^{-\sigma-2} u(x,t)\,\mathrm{d}x = 0,$$

because σ is assumed to satisfy Eq. (2.7). However, in our reasoning below, we have never used the fact that the quantity $\widetilde{A} = \int_{\mathbb{R}^n} |x|^{-\sigma} u(x,t)\,\mathrm{d}x$ is independent of time.

REMARK 2.3. *The denominator of the constant A defined in (2.13) was chosen in such a way to have $\int_{\mathbb{R}^n} |x|^{-\sigma} \left(u_0(x) - AV(x,1)\right)\mathrm{d}x = 0$ (see Theorem 5.1 below). Notice also that A is equal to the norm of the function $V(x,1)$ in the weighted space $L^2(\mathbb{R}^n, \mathrm{e}^{|x|^2/4}\,\mathrm{d}x)$ (see next remark).*

REMARK 2.4. *Our method of the proof of Theorem 2.1 does not work in the critical case, namely, for* $\lambda = \frac{(n-2)^2}{4}$. *Here, let us quote again the result by Vázquez and Zuazua [13, Theorem 10.3], who proved that for* $\lambda = \frac{(n-2)^2}{4}$ *and for any* $u_0 \in L^2(\mathbb{R}^n, \mathrm{e}^{|x|^2/4}\,\mathrm{d}x)$ *Cauchy problem (1.1)–(1.2) admits a solution u satisfying*

$$\lim_{t\to\infty} t^{\frac{1}{2}} \|u(\cdot,t) - AV(\cdot,t)\|_2 = 0.$$

In the proof, they worked in similarity variables and used an improved version of the Hardy inequality (1.3). Here, our reasoning is different and allows us to deal with a large class of initial conditions due to the optimal estimates of the fundamental solution of Eq. (1.1). However, we are not able to handle the critical value of the parameter $\lambda = \frac{(n-2)^2}{4}$.

REMARK 2.5. *It is worth pointing out that, by the assumption from Theorem 2.1, for* $\lambda \in \left(0, \frac{(n-2)^2}{4}\right)$ *we have* $\sigma > 0$, *hence, we consider the initial datum* u_0 *which is not too singular at zero. On the other hand, for* $\lambda < 0$, *we have* $\sigma < 0$, *hence, the initial condition* u_0 *has to decay at infinity sufficiently fast.*

REMARK 2.6. *The decay rate in (2.12) is optimal in the following sense. Using the self-similar form of* $V(x,t) = t^{\frac{\sigma-n}{2}} V\left(\frac{|x|}{\sqrt{t}}, 1\right)$ *and of the norm* $\|\cdot\|_{q,\varphi_\sigma(t)}$, *we obtain*

$$\|V(\cdot,t)\|_{q,\varphi_\sigma(t)} = t^{-\frac{n}{2}(1-\frac{1}{q})+\frac{\sigma}{2}} \|V(\cdot,1)\|_{q,\varphi_\sigma(1)}.$$

Applying this property of the self-similar solution, we may write

$$t^{\frac{n}{2}(1-\frac{1}{q})-\frac{\sigma}{2}} \|u(\cdot,t) - AV(\cdot,t)\|_{q,\varphi_\sigma(t)} = \|t^{\frac{n-\sigma}{2}} u(\sqrt{t}\cdot,t) - AV(\cdot,1)\|_{q,\varphi_\sigma(1)},$$

which means that $t^{\frac{n-\sigma}{2}} u(\sqrt{t}x, t)$ *converges toward* $AV(x,1)$ *in the norm* $\|\cdot\|_{q,\varphi_\sigma(1)}$.

In the next section, we discuss questions on the existence of solutions to initial value problem (1.1)–(1.2).

3. Existence of solutions

We begin our study of properties of solutions to (1.1)–(1.2) by deriving explicit solutions to Eq. (1.1).

PROPOSITION 3.1. *For every* $\lambda \leqslant \frac{(n-2)^2}{4}$, *all* $x \in \mathbb{R}^n \setminus \{0\}$ *and* $t > 0$, *Eq. (1.1) has two explicit solutions of the form*

$$u_1(x,t) = t^{\sigma-\frac{n}{2}} |x|^{-\sigma} \mathrm{e}^{-\frac{|x|^2}{4t}} \tag{3.1}$$

and

$$u_2(x,t) = t^{\frac{n}{2}-\sigma-2} |x|^{2-n+\sigma} \mathrm{e}^{-\frac{|x|^2}{4t}}, \tag{3.2}$$

where $\sigma = \sigma(\lambda)$ *is defined in (2.6).*

REMARK 3.2. *Notice that for* $\lambda = \frac{(n-2)^2}{4}$, *we have* $\sigma = \sigma(\lambda) = \frac{n-2}{2}$ *and, both formulas* (3.1) *and* (3.2) *reduce to one explicit solution of Eq.* (1.1) *in the following form*

$$u(x,t) = t^{-1}|x|^{-\frac{n-2}{2}} \mathrm{e}^{-\frac{|x|^2}{4t}} \tag{3.3}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$ *and* $t > 0$.

Proof of Proposition 3.1. One can check by a direct calculation that these functions satisfy Eq. (1.1) for all $x \in \mathbb{R}^n \setminus \{0\}$ and $t > 0$. Here, however, we sketch a reasoning which allowed us to find these formulas.

We look for a solution of Eq. (1.1) in the radial and self-similar form

$$u(x,t) = t^{-\alpha} U\left(\frac{|x|}{\sqrt{t}}\right)$$

for some $\alpha > 0$. Hence, it follows from direct calculations, that the function $U = U(|x|)$ satisfies the equation

$$U'' + \left(\frac{n-1}{r} + \frac{1}{2}r\right)U' + \left(\alpha + \frac{\lambda}{r^2}\right)U = 0. \tag{3.4}$$

Next, by a sequence of substitutions, we reduce Eq. (3.4) to the equation

$$(r^{m-1}g')' + \left(\frac{m}{4}r^{m-1} - \frac{1}{16}r^{m+1}\right)g = 0 \tag{3.5}$$

for some $m \in \mathbb{R}$ with the explicit solution $\mathrm{e}^{-\frac{r^2}{8}}$.

Indeed, first, we substitute the function $U(r) = \mathrm{e}^{-\frac{r^2}{8}} W(r)$ into Eq. (3.4) to get

$$(r^{n-1}W')' + \left(\left(\alpha - \frac{n}{4}\right)r^{n-1} - \frac{1}{16}r^{n+1} + \lambda r^{n-3}\right)W = 0.$$

Next, denoting $W(r) = r^b g(r)$ for some $b \in \mathbb{R}$ to be chosen later, we obtain

$$(r^{n+2b-1}g')' + \left(\left(\alpha - \frac{n}{4}\right)r^{n+2b-1} - \frac{1}{16}r^{n+2b+1} + \left(b^2 + (n-2)b + \lambda\right)r^{n+2b-3}\right) g = 0. \tag{3.6}$$

Here, we assume that the parameter b satisfies the quadratic equation (cf. Eq. (2.7))

$$b^2 + (n-2)b + \lambda = 0, \tag{3.7}$$

which, for $\lambda < \frac{(n-2)^2}{4}$, has two roots

$$b_1 = -\frac{n-2}{2} - \sqrt{\frac{(n-2)^2}{4} - \lambda} = -n - 2 + \sigma \quad \text{and}$$

$$b_2 = -\frac{n-2}{2} + \sqrt{\frac{(n-2)^2}{4} - \lambda} = -\sigma.$$

Choosing these numbers as b in (3.6) and substituting $\alpha = \frac{n+b}{2}$, it is easy to see that Eq. (3.6) takes the form (3.5) with $m = n + 2b$. Therefore, recalling the explicit solution $g(r) = \mathrm{e}^{-\frac{r^2}{8}}$ of (3.6) and inverting our substitutions, we obtain the solution (3.1) for $b = b_1$ and the solution (3.2) for $b = b_2$. □

Now, we recall a classical result on the existence of weak solutions to problem (1.1)–(1.2).

PROPOSITION 3.3. *For all* $\lambda \in \big(-\infty, \frac{(n-2)^2}{4}\big)$, $u_0 \in L^2(\mathbb{R}^n)$ *and* $T > 0$ *there exists a unique weak solution* u *of initial value problem* (1.1)–(1.2) *such that* $u \in C([0, T], L^2(\mathbb{R}^n))$ *and* $\nabla u \in L^2([0, T], L^2(\mathbb{R}^n))$.

Proof. If $\lambda \leqslant 0$, solutions of (1.1)–(1.2) satisfy the following *a priori* estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(\cdot, t)\|_2^2 + \|\nabla u(\cdot, t)\|_2^2 \leqslant 0. \tag{3.8}$$

For $\lambda \in \big(0, \frac{(n-2)^2}{4}\big)$, the Hardy inequality (1.3) implies the following energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(\cdot, t)\|_2^2 + 2\Big(1 - \lambda\frac{4}{(n-2)^2}\Big)\|\nabla u(\cdot, t)\|_2^2 \leqslant 0. \tag{3.9}$$

Thus, by the Galerkin method and *a priori* estimates (3.8) and (3.9), we obtain immediately following, for example, [5, Sect. 7.1.2] the existence and the uniqueness of a weak solution to problem (1.1)–(1.2). □

However, we can also study solutions of problem (1.1)–(1.2) via the semigroup theory. In this approach, we consider a sesquilinear form

$$a(u, v) = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, \mathrm{d}x - \lambda \int_{\mathbb{R}^n} |x|^{-2} uv \, \mathrm{d}x. \tag{3.10}$$

defined on $W^{1,2}(\mathbb{R}^n) \times W^{1,2}(\mathbb{R}^n)$. This form defines our operator $Hu = -\Delta u - \lambda |\cdot|^{-2}u$ by the formula $(Hz, v) = a(z, v)$ on the domain $\mathcal{D}(H) = \{z \in W^{1,2}(\mathbb{R}^n) : |a(z, v)| \leqslant C\|v\|_2, v \in W^{1,2}(\mathbb{R}^n)\}$.

PROPOSITION 3.4. *The operator* $-H$ *defined via the sesquilinear form* (3.10) *is an infinitesimal generator of strongly continuous semigroup of linear operators* $\{\mathrm{e}^{-tH}\}_{t\geqslant 0}$ *on* $L^2(\mathbb{R}^n)$.

Proof. Here, it suffices to show (see, for example, either [3, Prop. 1.1] or in [11, Prop. 1.51]) that

(1) the sesquilinear form $a(u, v)$ defined in (3.10) is bounded on $W^{1,2}(\mathbb{R}^n)$, which is a direct consequence of the Hardy inequality (1.3);
(2) for some $\alpha > 0$ and $\lambda_0 \in \mathbb{R}$ we have

$$\alpha\|u\|_{W^{1,2}}^2 \leqslant \mathrm{Re}\, a(u, u) + \lambda_0\|u\|_2^2, \tag{3.11}$$

which results again from (1.3) (cf. (3.8), (3.9)). □

Assumptions (1) and (2) from the above proof imply, in fact, that $\{e^{-tH}\}_{t\geqslant 0}$ is an analytic semigroup of linear operators on $L^2(\mathbb{R}^n)$. However, in the following, we do not need this additional property of $\{e^{-tH}\}_{t\geqslant 0}$.

REMARK 3.5. *It follows from Proposition 3.4 that the weak solution from Proposition 3.3 can be written in the form* $u(x,t) = e^{-tH}u_0(x)$.

We have recalled these well-known facts in order to explain that the function given by formula (3.1) is a semigroup solution of (1.1)–(1.2). From now on, we denote this explicit solution by

$$V(x,t) = t^{\sigma-\frac{n}{2}}|x|^{-\sigma}e^{-\frac{|x|^2}{4t}}. \tag{3.12}$$

THEOREM 3.6. *The function* $V(x,t+1)$ *is a solution to Eq.* (1.1) *with the initial condition* $u_0(x) = V(x,1)$ *in the sense of Proposition* 3.3. *In other words,* $V(x,t+1) = e^{-tH}V(x,1)$.

Proof. For $\lambda < \frac{(n-2)^2}{4}$, we have $2\sigma < n$, hence, $V(\cdot,1) \in L^2(\mathbb{R}^n)$. Hence, in the view of Proposition 3.3, it is enough to check that $V(\cdot,\cdot+1) \in C([0,T],L^2(\mathbb{R}^n))$ and $\nabla V(\cdot,\cdot+1) \in L^2([0,T],L^2(\mathbb{R}^n))$. This is, however, the consequence of the fact that the function $V(x,t+1)$ is rapidly decreasing as $|x| \to \infty$ with singularity at $x=0$ like $|x|^{-\sigma}$ with $2\sigma < n$. The gradient of the function V can be handled in the same way, because $\nabla V(x,t)$ is singular at the origin like $|x|^{-\sigma-1}$, which is locally in $L^2(\mathbb{R}^n)$, since $2\sigma < n-2$, (cf. (2.6) and (2.8)). □

4. Properties of fundamental solution

We recall two-sided estimates of the fundamental solution of Eq. (1.1), which was found independently by Liskevich and Sobol [8, Remarks at the end of Sec. 1] and Milman and Semenov [9, Theorem 1] (see also [10,14,15]).

THEOREM 4.1. *Assume that* $\lambda \in \left(-\infty, \frac{(n-2)^2}{4}\right)$. *Then, the semigroup of linear operators* e^{-tH} *from Proposition* 3.4 *can be written as the integral operator with a kernel* $K(x,y,t)$*, namely,*

$$e^{-tH}u_0(x) = \int_{\mathbb{R}^n} K(x,y,t)u_0(y)\,dy. \tag{4.1}$$

Moreover, there exist positive constants $C_1, C_2 > 0$ *and* $c_1, c_2 > 1$*, such that for all* $t > 0$ *and all* $x, y \in \mathbb{R}^n \setminus \{0\}$

$$\begin{aligned} C_1\varphi_\sigma(x,t)\,\varphi_\sigma(y,t)\,G(x-y,c_1t) &\leqslant K(x,y,t)\\ &\leqslant C_2\varphi_\sigma(x,t)\,\varphi_\sigma(y,t)\,G(x-y,c_2t), \end{aligned} \tag{4.2}$$

where $\sigma = \sigma(\lambda)$ *is given by* (2.6)*, the functions* φ_σ *are defined in* (2.9) *and* $G(x,t) = \left(4\pi t\right)^{-n/2}\exp(-|x|^2/4t)$ *is the heat kernel.*

REMARK 4.2. *In fact, the authors of* [8,9] *used more regular weight functions* $\Phi_\sigma \in C^2(\mathbb{R}^n \setminus \{0\})$*, namely,*

$$\Phi_\sigma(x,t) = \begin{cases} \left(\frac{\sqrt{t}}{|x|}\right)^\sigma & \text{if } |x| \leqslant \sqrt{t}, \\ C & \text{if } |x| \geqslant 2\sqrt{t} \end{cases}$$

satisfying $\min\{1, C\} \leqslant \Phi_\sigma(x,t) \leqslant \max\{1, C\}$ *for* $\sqrt{t} \leqslant |x| \leqslant 2\sqrt{t}$ *with a suitable constant* C *and such that* Φ_σ *is sufficiently regular for* $\sqrt{t} \leqslant |x| \leqslant 2\sqrt{t}$*. It can be checked directly that there exist positive constants* C_1 *and* C_2 *for which the inequalities*

$$C_1\varphi_\sigma(x,t) \leqslant \Phi_\sigma(x,t) \leqslant C_2\varphi_\sigma(x,t)$$

hold true, where φ_σ *is defined by* (2.9)*. By this reason, we are allowed to use the weights* φ_σ *instead of* Φ_σ *in estimates* (4.2)*.*

Estimates (4.2) of the fundamental solution allow us to consider a class of initial conditions, which is different than $L^2(\mathbb{R}^n)$. In particular, by direct calculations involving the estimates of the kernel K from Theorem 4.1 and the definition of the weight functions φ_σ, we obtain

$$\|\mathrm{e}^{-tH} f\|_{q,\varphi_\sigma(t)}^q \leqslant C\|G(c_2t) * f\varphi_\sigma(t)\|_q^q + Ct^\sigma \|G(c_2t) * f\varphi_\sigma(t)\|_\infty^q \int_{|x|\leqslant\sqrt{t}} |x|^{-2\sigma}\,\mathrm{d}x.$$

Next, applying the Young inequality for the convolution, the scaling property of the heat kernel $G = G(x,t)$, and the fact that $2\sigma < n$, we get the following inequality

$$\|\mathrm{e}^{-tH} f\|_{q,\varphi_\sigma(t)} \leqslant Ct^{-\frac{n}{2}(1-\frac{1}{q})}\|f\|_{1,\varphi_\sigma(t)} \tag{4.3}$$

that is valid for every $q \in [1, \infty]$, all $t > 0$, each measurable function f such that $\|f\|_{1,\varphi_\sigma(t)}$ is finite and a constant $C = C(q, 1)$ independent of t and f. Let us emphasize, that this kind of inequalities have been systematically used in [9] to derive the kernel estimates (4.2).

Next, it is worth pointing out that the operator e^{-tH} is symmetric, thus, its kernel $K(x, y, t)$ is also symmetric with respect to x and y. Moreover, the function $K(x, y, t)$ in representation (4.1) is unique. Now, let us discuss its continuity.

LEMMA 4.3. *For every* $\lambda \in \left(-\infty, \frac{(n-2)^2}{4}\right)$ *and each* $t > 0$*, the function*

$$\tilde{S}(x, y, t) = |x|^\sigma K(x, y, t)|y|^\sigma \tag{4.4}$$

defined for all $x, y \in \mathbb{R}^n \setminus \{0\}$ *can be extended to a continuous function (denoted also by* $\tilde{S}$*) with respect to* $x, y \in \mathbb{R}^n$*.*

Proof. First, notice that the kernel $K(\cdot, y, t)$ is a weak solution to Eq. (1.1). Thus, for $\lambda \in \left(0, \frac{(n-2)^2}{4}\right)$, one can apply results from [10] to $K(x, y, t)$. Let us briefly review those arguments. Notice that the parameter λ_+ in [10, Theorem 3.8.] corresponds to -2σ in our case. Introducing the transformation $v = \frac{u}{\phi}$, where $\phi(x) = |x|^{-\sigma}$ is a positive weak solution of $\Delta\phi + \frac{\lambda}{|x|^2}\phi = 0$, Eq. (1.1) becomes

$$v_t = \frac{1}{\phi^2}\operatorname{div}(\phi^2\nabla v). \tag{4.5}$$

The Moser iteration technique applied to Eq. (4.5) allowed the authors of [10] to show the Hölder continuity of solutions to Eq. (4.5) (see [10, Theorem 3.8] for more details). This implies that $\tilde{S}(x, y, t)$ is a continuous function with respect to x for each $y \in \mathbb{R}^n$ and by symmetry, a continuous function with respect to y for each $x \in \mathbb{R}^n$. The same reasoning can also be directly applied in the case $\lambda \leqslant 0$ following [10] (see also [2]).

Now, we show the continuity of $\tilde{S} = \tilde{S}(x, y, t)$ with respect to both variables (x, y) and, for simplicity of the exposition, we consider $(x, y) = (0, 0)$. The proof for $(x, y) \neq (0, 0)$ is completely analogous. First, let us notice that the function $\tilde{S}$ satisfies the usual Chapman–Kolmogorov equality, because it is a fundamental solution to Eq. (4.5) with $\phi(x) = |x|^{-\sigma}$. We fix $\varepsilon > 0$ and take $x, y \in \mathbb{R}^n$ such that $|x| < \delta$ and $|y| < \delta$ for some $\delta > 0$ and we apply the Chapman–Kolmogorov equality, we get

$$\begin{aligned}\tilde{S}(x, y, t) - \tilde{S}(0, 0, t) &= \tilde{S}(x, y, t) - \tilde{S}(x, 0, t) + \tilde{S}(x, 0, t) - \tilde{S}(0, 0, t)\\ &= \int_{\mathbb{R}^n} \tilde{S}\Big(x, z, \frac{t}{2}\Big)\Big[\tilde{S}\Big(z, y, \frac{t}{2}\Big) - \tilde{S}\Big(z, 0, \frac{t}{2}\Big)\Big]|z|^{-2\sigma}\,\mathrm{d}z\\ &\quad + \int_{\mathbb{R}^n} \tilde{S}\Big(z, 0, \frac{t}{2}\Big)\Big[\tilde{S}\Big(x, z, \frac{t}{2}\Big) - \tilde{S}\Big(0, z, \frac{t}{2}\Big)\Big]|z|^{-2\sigma}\,\mathrm{d}z.\end{aligned}$$

Since the functions $\tilde{S}(x, z, t/2)$ and $\tilde{S}(z, 0, t/2)$ have the Gaussian estimates (cf. Theorem 4.1), using the elementary inequality $|z - x|^2 \geqslant |z|^2/2 - \delta^2$, we obtain

$$\begin{aligned}\Big|\tilde{S}(x, y, t) - \tilde{S}(0, 0, t)\Big| &\leqslant C\int_{\mathbb{R}^n} \mathrm{e}^{-\frac{|z|^2}{2tc}}|z|^{-2\sigma}\Big[\tilde{S}\Big(z, y, \frac{t}{2}\Big) - \tilde{S}\Big(z, 0, \frac{t}{2}\Big)\Big]\,\mathrm{d}z\\ &\quad + C\int_{\mathbb{R}^n} \mathrm{e}^{-\frac{|z|^2}{2tc}}|z|^{-2\sigma}\Big[\tilde{S}\Big(x, z, \frac{t}{2}\Big) - \tilde{S}\Big(0, z, \frac{t}{2}\Big)\Big]\,\mathrm{d}z.\end{aligned}$$

Now, the continuity of the function $\tilde{S}(z, y, t/2)$ with respect to y for every $z \in \mathbb{R}^n$ and $\tilde{S}(x, z, t/2)$ with respect to x for every $z \in \mathbb{R}^n$ combining with the Lebesgue dominated convergence theorem, allows us to find δ such that

$$\Big|\tilde{S}(x, y, t) - \tilde{S}(0, 0, t)\Big| < \varepsilon.$$

This completes the proof of Lemma 4.3. □

THEOREM 4.4. *The kernel of the operator* e^{-tH} *provided by Theorem 4.2 has the self-similar form, namely,*

$$K(x, y, t) = t^{-\frac{n}{2}}K\Big(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}, 1\Big) \tag{4.6}$$

for all $t > 0$ *and* $x, y \in \mathbb{R}^n \setminus \{0\}$.

Proof. Let us note that if the function $u = u(x,t)$ is the solution of the problem (1.1)–(1.2), then the function $u^\alpha = u^\alpha(x,t) = u(\alpha x, \alpha^2 t)$ is also a solution of (1.1) for any $\alpha > 0$ with the initial datum $u_0^\alpha(x) = u_0(\alpha x)$. However, by Theorem 4.1, every semigroup solution of problem (1.1)–(1.2) has the form

$$u(x,t) = \int_{\mathbb{R}^n} K(x,y,t) u_0(y)\, \mathrm{d}y,$$

hence

$$u^\alpha(x,t) = \int_{\mathbb{R}^n} K(x,y,t) u_0^\alpha(y)\, \mathrm{d}y = \int_{\mathbb{R}^n} K(x,y,t) u_0(\alpha y)\, \mathrm{d}y.$$

Substituting $\alpha y = \bar{y}$ and recalling the definition of u^α, we get

$$u(\alpha x, \alpha^2 t) = \alpha^{-n} \int_{\mathbb{R}^n} K\left(x, \frac{\bar{y}}{\alpha}, t\right) u_0(\bar{y})\, \mathrm{d}\bar{y}.$$

Introducing the new notation $\bar{x} = \alpha x$ and $\bar{t} = \alpha^2 t$, we obtain

$$u(\bar{x}, \bar{t}) = \alpha^{-n} \int_{\mathbb{R}^n} K\left(\frac{\bar{x}}{\alpha}, \frac{\bar{y}}{\alpha}, \frac{\bar{t}}{\alpha^2}\right) u_0(\bar{y})\, \mathrm{d}\bar{y}$$

for all $\bar{x} \in \mathbb{R}^n \setminus \{0\}$ and $t > 0$. From the uniqueness of the kernel $K(x,y,t)$ and its continuity for $x \neq 0$ and $y \neq$ (by Lemma 4.3), we infer the equality

$$K(x,y,t) = \alpha^{-n} K\left(\frac{x}{\alpha}, \frac{y}{\alpha}, \frac{t}{\alpha^2}\right)$$

for all $x, y \in \mathbb{R}^n \setminus \{0\}$, $t > 0$ and $\alpha > 0$. Substituting $\alpha = \sqrt{t}$, we complete the proof of (4.6). □

Using the self-similar form of $K(x,y,t)$ and its continuity stated in Lemma 4.3, we prove two technical lemmas, which will be needed to obtain our main result.

LEMMA 4.5. *For every* $x, y \in \mathbb{R}^n$ *and* $t > 0$*, define*

$$S_\infty(x,y,t) = \varphi_\sigma^{-1}(x,t) K(x,y,t) \varphi_\sigma^{-1}(y,t),$$

where $K(x,y,t)$ *is the fundamental solution from Theorem* 4.1 *and* $\varphi_\sigma(x,t)$ *is the weight function stated in* (2.9). *Then, for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$t^{\frac{n}{2}} \sup_{\substack{x \in \mathbb{R}^n \\ |y| \leqslant \delta\sqrt{t}}} \Big| S_\infty(x,y,t) - S_\infty(x,0,t) \Big| < \varepsilon \tag{4.7}$$

for all $t > 0$.

Proof. By Lemma 4.3 and the explicit form of the function φ_σ, we immediately obtain that $S_\infty(x, y, t)$ can be extended to a continuous function (also denoted by S) for all $x, y \in \mathbb{R}^n$ and $t > 0$.

Let $\varepsilon > 0$. Using the self-similar form of kernel $K(x, y, t)$ and of the weight $\varphi_\sigma(x, t) = \varphi_\sigma\big(\frac{x}{\sqrt{t}}, 1\big)$, we get

$$\begin{aligned}
t^{\frac{n}{2}} \sup_{\substack{x\in\mathbb{R}^n \\ |y|\leqslant\delta\sqrt{t}}} &\Big|S_\infty(x, y, t) - S_\infty(x, 0, t)\Big| \\
&= \sup_{\substack{x\in\mathbb{R}^n \\ |y|\leqslant\delta\sqrt{t}}} \left|S_\infty\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}, 1\right) - S_\infty\left(\frac{x}{\sqrt{t}}, 0, 1\right)\right| \\
&= \sup_{\substack{x\in\mathbb{R}^n \\ |y|\leqslant\delta}} \Big|S_\infty(x, y, 1) - S_\infty(x, 0, 1)\Big|.
\end{aligned}$$

First, we choose $R > 1$ so large that

$$\sup_{\substack{|x|\geqslant R \\ |y|\leqslant\delta}} \Big|S_\infty(x, y, 1) - S_\infty(x, 0, 1)\Big| < \frac{\varepsilon}{2},$$

which is possible due to the Gaussian estimates of the function $S_\infty(x, y, 1)$, cf. the definition of $S_\infty(x, y, t)$ and inequalities (4.2).

Next, for fixed $R > 1$, the uniform continuity of function $S_\infty(x, y, 1)$ for $|x| \leqslant R$ and $|y| \leqslant \delta$ (cf. Lemma 4.3) allows us to find δ such that

$$\sup_{\substack{|x|\leqslant R \\ |y|\leqslant\delta}} \Big|S_\infty(x, y, 1) - S_\infty(x, 0, 1)\Big| < \frac{\varepsilon}{2}.$$

This completes the proof of inequality (4.7). □

LEMMA 4.6. *Let us define*

$$S_1(x, y, t) = \varphi_\sigma(x, t)K(x, y, t)\varphi_\sigma^{-1}(y, t) = S_\infty(x, y, t)\varphi_\sigma^2(x, t),$$

where $S_\infty(x, y, t)$ *is the function from Lemma 4.5 and* $\varphi_\sigma(x, t)$ *is the weight stated in* (2.9). *Then for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\sup_{|y|\leqslant\delta\sqrt{t}} \|S_1(\cdot, y, t) - S_1(\cdot, 0, t)\|_{L^1(\mathbb{R}^n)} < \varepsilon \tag{4.8}$$

for all $t > 0$.

Proof. We fix $\varepsilon > 0$. In view of the self-similar form of the kernel $K(x, y, t)$, the function $S_1(x, y, t)$ has the same scaling property. Hence, we obtain

$$\begin{aligned}
&\sup_{|y|\leqslant\delta\sqrt{t}} \|S_1(\cdot, y, t) - S_1(\cdot, 0, t)\|_{L^1(\mathbb{R}^n)} \\
&= t^{-\frac{n}{2}} \sup_{|y|\leqslant\delta\sqrt{t}} \int_{\mathbb{R}^n} \left| S_1\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}, 1\right) - S_1\left(\frac{x}{\sqrt{t}}, 0, 1\right)\right| \mathrm{d}x \\
&= \sup_{|w|\leqslant\delta} \|S_1(\cdot, w, 1) - S_1(\cdot, 0, 1)\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

First, we can choose $R > 1$ so large that

$$\begin{aligned}
&\sup_{|w|\leqslant\delta} \int_{|z|\geqslant R} |S_1(z, w, 1) - S_1(z, 0, 1)|\, \mathrm{d}z \\
&\leqslant \sup_{|w|\leqslant\delta} \int_{|z|\geqslant R} \big(|S_1(z, w, 1)| + |S_1(z, 0, 1)|\big)\, \mathrm{d}z \\
&\leqslant C_2 \sup_{|w|\leqslant\delta} \int_{|z|\geqslant R} \varphi_\sigma^2(z, 1)[G(z - w, c_2) + G(z, c_2)]\, \mathrm{d}z < \frac{\varepsilon}{2},
\end{aligned}$$

since, by Theorem 4.1, we have the Gaussian estimates of the function $S_\infty(z, w, 1)$ and the function $\varphi_\sigma(z, 1)$ is equal to 1 for $|z| \geqslant R > 1$.

Next, for fixed $R > 1$ choosing sufficiently small $\delta > 0$, we have by Lemma 4.5

$$\begin{aligned}
&\sup_{|w|\leqslant\delta} \int_{|z|\leqslant R} |S_1(z, w, 1) - S_1(z, 0, 1)|\, \mathrm{d}z \\
&= \sup_{|w|\leqslant\delta} \int_{|z|\leqslant R} |S_\infty(z, w, 1) - S_\infty(z, 0, 1)|\varphi_\sigma^2(z, 1)\, \mathrm{d}z \\
&\leqslant C(R) \sup_{\substack{|w|\leqslant\delta \\ |z|\leqslant R}} |S_\infty(z, w, 1) - S_\infty(z, 0, 1)| < C(R)\varepsilon,
\end{aligned}$$

where the constant $C(R) = \int_{|z|\leqslant R} \varphi_\sigma^2(z, 1)\, \mathrm{d}z$ is finite, because $|z|^{-2\sigma}$ is locally integrable by the properties of σ from (2.8). □

We conclude this section by an estimate involving the weighted L^1-norm.

LEMMA 4.7. *For all $f \in L^1(\mathbb{R}^n)$ satisfying $|\cdot|^{-\sigma} f \in L^1(\mathbb{R}^n)$, we have*

i) if $0 \leqslant \lambda < \frac{(n-2)^2}{4}$, then

$$\|f\|_{1,\varphi_\sigma(t)} \leqslant \|f\|_{L^1(\mathbb{R}^n)} + t^{\frac{\sigma}{2}} \||\cdot|^{-\sigma} f\|_{L^1(\mathbb{R}^n)}, \tag{4.9}$$

ii) if $\lambda < 0$, then

$$\|f\|_{1,\varphi_\sigma(t)} \leqslant t^{\frac{\sigma}{2}} \||\cdot|^{-\sigma} f\|_{L^1(\mathbb{R}^n)} \tag{4.10}$$

for all $t > 0$.

Proof. This is the immediately consequence of the definition of the weighted L^1-norm and estimates $\varphi_\sigma(x, t) \leqslant 1 + t^{\frac{\sigma}{2}}|x|^{-\sigma}$ for $\lambda \geqslant 0$ and $\varphi_\sigma(x, t) \leqslant t^{\frac{\sigma}{2}}|x|^{-\sigma}$ for $\lambda \leqslant 0$. □

5. Self-similar asymptotics—proof of Theorem 2.1

Our main result stated in Theorem 2.1 on the large-time asymptotics of solutions to (1.1)–(1.2) will be a direct consequence of the following property of the semigroup e^{-tH} of linear operators constructed in Proposition 3.4.

THEOREM 5.1. *Assume that* $w_0 \in L^1(\mathbb{R}^n)$ *satisfies* $\int_{\mathbb{R}^n} |x|^{-\sigma}|w_0(x)|\,dx < \infty$ *and*

$$\int_{\mathbb{R}^n} w_0(x)|x|^{-\sigma}\,dx = 0. \tag{5.1}$$

Then, for every $1 \leqslant q \leqslant \infty$

$$\lim_{t\to\infty} t^{\frac{n}{2}(1-\frac{1}{q})-\frac{\sigma}{2}} \|e^{-tH}w_0\|_{q,\varphi_\sigma(t)} = 0, \tag{5.2}$$

where for $q = \infty$ *the expression* $\frac{1}{q}$ *is understood as 0.*

Proof. For simplicity of the exposition, we divide this proof into a series of steps. First, we consider a compactly supported function ψ such that $|\cdot|^{-\sigma}\psi \in L^1(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} |x|^{-\sigma}\psi(x)\,dx = 0. \tag{5.3}$$

Step 1. Convergence in weighted L^∞.
If $y \in \operatorname{supp}\,\psi$, we assume that $\varphi_\sigma(y,t) = \big(\frac{\sqrt{t}}{|y|}\big)^\sigma$ for sufficiently large $t > 0$. Hence, using the definition of weight function φ_σ and norm $\|\cdot\|_{\infty,\varphi_\sigma(t)}$ for large $t > 0$, we have

$$\begin{aligned} I(t) &\equiv t^{\frac{n-\sigma}{2}} \|e^{-tH}\psi\|_{\infty,\varphi_\sigma(t)} \\ &\leqslant t^{\frac{n-\sigma}{2}} \sup_{x\in\mathbb{R}^n} \left|\varphi_\sigma^{-1}(x,t)\int_{\mathbb{R}^n} K(x,y,t)\varphi_\sigma^{-1}(y,t)\varphi_\sigma(y,t)\psi(y)\,dy\right| \\ &= t^{\frac{n}{2}} \sup_{x\in\mathbb{R}^n} \left|\int_{\mathbb{R}^n} S_\infty(x,y,t)|y|^{-\sigma}\psi(y)\,dy\right|. \end{aligned}$$

Applying (5.3), we get

$$I(t) = t^{\frac{n}{2}} \sup_{x\in\mathbb{R}^n} \left|\int_{\mathbb{R}^n} \big[S_\infty(x,y,t) - S_\infty(x,0,t)\big]|y|^{-\sigma}\psi(y)\,dy\right|.$$

Now, we fix $\varepsilon > 0$ and we observe that for sufficiently large t, since the function ψ has a compact support, we may write

$$I(t) = t^{\frac{n}{2}} \sup_{x\in\mathbb{R}^n} \left|\int_{|y|\leqslant\delta\sqrt{t}} \big[S_\infty(x,y,t) - S_\infty(x,0,t)\big]|y|^{-\sigma}\psi(y)\,dy\right|$$

with fixed $\delta > 0$. By Lemma 4.5, we can choose $\delta > 0$ such that

$$\begin{aligned} I(t) &= t^{\frac{n}{2}} \sup_{x\in\mathbb{R}^n} \left|\int_{|y|\leqslant\delta\sqrt{t}} \big[S_\infty(x,y,t) - S_\infty(x,0,t)\big]|y|^{-\sigma}\psi(y)\,dy\right| \\ &\leqslant t^{\frac{n}{2}} \sup_{\substack{x\in\mathbb{R}^n \\ |y|\leqslant\delta\sqrt{t}}} \big|S_\infty(x,y,t) - S_\infty(x,0,t)\big| \int_{|y|\leqslant\delta\sqrt{t}} |y|^{-\sigma}|\psi(y)|\,dy < \varepsilon. \end{aligned}$$

Step 2. Convergence in weighted L^1.
Now, let $q = 1$. Using the definitions of $\|\cdot\|_{1,\varphi_\sigma(t)}$ and the function S_1 from Lemma 4.6, we arrive at

$$J(t) \equiv t^{-\frac{\sigma}{2}} \|\mathrm{e}^{-tH}\psi\|_{1,\varphi_\sigma(t)} = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} S_1(x,y,t)\psi(y)|y|^{-\sigma}\, \mathrm{d}y \right| \mathrm{d}x.$$

Applying condition (5.3), we get

$$J(t) \leqslant \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big|S_1(x,y,t) - S_1(x,0,t)\big||\psi(y)||y|^{-\sigma}\, \mathrm{d}y\, \mathrm{d}x.$$

We fix $\delta > 0$ and we notice that

$$J(t) \leqslant \int_{\mathbb{R}^n}\int_{|y|\leqslant\delta\sqrt{t}} \big|S_1(x,y,t) - S_1(x,0,t)\big||\psi(y)||y|^{-\sigma}\, \mathrm{d}y\, \mathrm{d}x,$$

for sufficiently large t, since the function ψ has a compact support. We deal with the term on the right-hand side in the most direct way

$$\begin{aligned} J(t) &\leqslant \int_{\mathbb{R}^n}\int_{|y|\leqslant\delta\sqrt{t}} \big|S_1(x,y,t) - S_1(x,0,t)\big||\psi(y)||y|^{-\sigma}\, \mathrm{d}y\, \mathrm{d}x \\ &\leqslant \sup_{|y|\leqslant\delta\sqrt{t}} \|S_1(\cdot,y,t) - S_1(\cdot,0,t)\|_{L^1(\mathbb{R}^n)} \int_{\mathbb{R}^n} |y|^{-\sigma}|\psi(y)|\, \mathrm{d}y. \end{aligned}$$

From Lemma 4.6, we claim that

$$\sup_{|y|\leqslant\delta\sqrt{t}} \|S_1(\cdot,y,t) - S_1(\cdot,0,t)\|_{L^1(\mathbb{R}^n)} < \varepsilon$$

provided $\delta > 0$ is sufficiently small. Hence,

$$J(t) \leqslant \varepsilon \||\cdot|^{-\sigma}\psi\|_{L^1(\mathbb{R}^n)} = C\varepsilon$$

since the function $|y|^{-\sigma}|\psi(y)|$ is integrable. *Step 3. Convergence in weighted L^q.* To show the L^q-estimates for $1 < q < \infty$, we use the definition of the norm $\|\cdot\|_{q,\varphi_\sigma(t)}$ as follows:

$$\begin{aligned} &t^{\frac{n}{2}(1-\frac{1}{q})-\frac{\sigma}{2}} \|\mathrm{e}^{-tH}\psi\|_{q,\varphi_\sigma(t)} \\ &= t^{\frac{n}{2}(1-\frac{1}{q})-\frac{\sigma}{2}} \left(\int_{\mathbb{R}^n} \big|\mathrm{e}^{-tH}\psi(x)\varphi_\sigma^{-1}(x,t)\big|^{q-1} \big|\mathrm{e}^{-tH}\psi(x)\varphi_\sigma(x,t)\big|\, \mathrm{d}x \right)^{\frac{1}{q}} \\ &\leqslant \left[t^{\frac{n-\sigma}{2}} \|\mathrm{e}^{-tH}\psi\|_{\infty,\varphi_\sigma(t)}\right]^{1-\frac{1}{q}} \left[t^{-\frac{\sigma}{2}} \|\mathrm{e}^{-tH}\psi\|_{1,\varphi_\sigma(t)}\right]^{\frac{1}{q}}. \end{aligned}$$

Factors on the right-hand side converges to zero as $t \to \infty$ by *Step 1* and *Step 2* of this proof. *Step 4. General initial datum.* Let us complete the proof of (5.2) for every w_0 satisfying the assumptions of Theorem 5.1. Notice that for every $R > 0$ there exists a constant $c_R \in \mathbb{R}$ such that the compactly supported function $\psi(x) =$

$(w_0(x)-c_R)\mathbb{1}_{|x|\leqslant R}(x)$ satisfies $\int_{\mathbb{R}^n}|x|^{-\sigma}\psi(x)\,\mathrm{d}x = 0$. Indeed, since $w_0(x)-\psi(x) = w_0(x)\mathbb{1}_{|x|>R}(x) + c_R\mathbb{1}_{|x|\leqslant R}(x)$, we choose c_R in such a way to have

$$c_R\int_{|x|\leqslant R}|x|^{-\sigma}\,\mathrm{d}x = \int_{|x|\leqslant R}|x|^{-\sigma}w_0(x)\,\mathrm{d}x.$$

Consequently, we obtain the estimate

$$\begin{aligned}\|(w_0-\psi)|\cdot|^{-\sigma}\|_{L^1(\mathbb{R}^n)} &\leqslant \int_{|x|>R}|x|^{-\sigma}|w_0(x)|\,\mathrm{d}x + |c_R|\int_{|x|\leqslant R}|x|^{-\sigma}\,\mathrm{d}x\\ &\leqslant \int_{|x|>R}|x|^{-\sigma}|w_0(x)|\,\mathrm{d}x + \left|\int_{|x|\leqslant R}|x|^{-\sigma}w_0(x)\,\mathrm{d}x\right|,\end{aligned}$$

where the right-hand side tends to zero as $R\to\infty$, due to assumption (5.1) and the condition $|\cdot|^{-\sigma}w_0 \in L^1(\mathbb{R}^n)$. In other words, for every $\varepsilon>0$, we can choose $R>0$ so large that

$$\|(w_0-\psi)|\cdot|^{-\sigma}\|_{L^1(\mathbb{R}^n)} < \varepsilon.$$

Now, using the triangle inequality and estimate (4.3), we obtain for every $1<q\leqslant\infty$

$$t^{\frac{n}{2}(1-\frac{1}{q})-\frac{\sigma}{2}}\|\mathrm{e}^{-tH}w_0\|_{q,\varphi_\sigma(t)} \leqslant t^{\frac{n}{2}(1-\frac{1}{q})-\frac{\sigma}{2}}\|\mathrm{e}^{-tH}\psi\|_{q,\varphi_\sigma(t)} + Ct^{-\frac{\sigma}{2}}\|w_0-\psi\|_{1,\varphi_\sigma(t)}.$$

Hence, for $0<\lambda<\frac{(n-2)^2}{4}$, using inequality (4.9), we have

$$\begin{aligned}&t^{\frac{n}{2}(1-\frac{1}{q})-\frac{\sigma}{2}}\|\mathrm{e}^{-tH}w_0\|_{q,\varphi_\sigma(t)}\\ &\quad\leqslant t^{\frac{n}{2}(1-\frac{1}{q})-\frac{\sigma}{2}}\|\mathrm{e}^{-tH}\psi\|_{q,\varphi_\sigma(t)}\\ &\qquad+Ct^{-\frac{\sigma}{2}}\|w_0-\psi\|_{L^1(\mathbb{R}^n)} + C\|(w_0-\psi)|\cdot|^{-\sigma}\|_{L^1(\mathbb{R}^n)}\\ &\quad\leqslant t^{\frac{n}{2}(1-\frac{1}{q})-\frac{\sigma}{2}}\|\mathrm{e}^{-tH}\psi\|_{q,\varphi_\sigma(t)} + Ct^{-\frac{\sigma}{2}}\|w_0-\psi\|_{L^1(\mathbb{R}^n)} + C\varepsilon.\end{aligned}$$

Since the first term on the right-hand side converges to zero as $t\to\infty$ by *Steps 1–3* of this proof (recall that ψ has a compact support) and the second one also converges to zero as $t\to\infty$, because $\sigma>0$ and $w_0-\psi\in L^1(\mathbb{R}^n)$, we get

$$\limsup_{t\to\infty} t^{\frac{n}{2}(1-\frac{1}{q})-\frac{\sigma}{2}}\|\mathrm{e}^{-tH}w_0\|_{q,\varphi_\sigma(t)} \leqslant C\varepsilon \text{ for every } \varepsilon>0. \tag{5.4}$$

If $\lambda<0$, applying inequality (4.10) from Lemma 4.7, we arrive at

$$\begin{aligned}&t^{\frac{n}{2}(1-\frac{1}{q})-\frac{\sigma}{2}}\|\mathrm{e}^{-tH}w_0\|_{q,\varphi_\sigma(t)}\\ &\quad\leqslant t^{\frac{n}{2}(1-\frac{1}{q})-\frac{\sigma}{2}}\|\mathrm{e}^{-tH}\psi\|_{q,\varphi_\sigma(t)} + C\|(w_0-\psi)|\cdot|^{-\sigma}\|_{L^1(\mathbb{R}^n)}\\ &\quad\leqslant t^{\frac{n}{2}(1-\frac{1}{q})-\frac{\sigma}{2}}\|\mathrm{e}^{-tH}\psi\|_{q,\varphi_\sigma(t)} + C\varepsilon \text{ for every } \varepsilon>0.\end{aligned}$$

The first term on the right-hand side converges to zero when $t\to\infty$, again by *Steps 1–3* of this proof.

This finishes the proof of Theorem 5.1, because $\varepsilon>0$ can be arbitrarily small. □

Proof of Theorem 2.1. We first observe that the function $w_0(x) = u_0(x) - AV(x,1)$ satisfies the assumptions of Theorem 5.1, because

$$\int_{\mathbb{R}^n} |x|^{-\sigma} w_0(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} |x|^{-\sigma} u_0(x)\,\mathrm{d}x - \frac{\int_{\mathbb{R}^n} |x|^{-\sigma} u_0(x)\,\mathrm{d}x}{\int_{\mathbb{R}^n} |x|^{-2\sigma} \mathrm{e}^{-\frac{|x|^2}{4}}\,\mathrm{d}x} \times \int_{\mathbb{R}^n} |x|^{-2\sigma} \mathrm{e}^{-\frac{|x|^2}{4}}\,\mathrm{d}x = 0.$$

Now, the semigroup property $\mathrm{e}^{-tH} V(x,1) = V(x,t+1)$ (see Remark 3.5) together with (5.2) leads us to the following equality

$$\lim_{t\to\infty} t^{\frac{n}{2}(1-\frac{1}{q})-\frac{\sigma}{2}} \|\mathrm{e}^{-tH} u_0(\cdot) - AV(\cdot,t+1)\|_{q,\varphi_\sigma(t)} = \lim_{t\to\infty} t^{\frac{n}{2}(1-\frac{1}{q})-\frac{\sigma}{2}} \|\mathrm{e}^{-tH}\big(u_0(\cdot) - AV(\cdot,1)\big)\|_{q,\varphi_\sigma(t)} = 0.$$

To complete the proof of (2.12), we will show that

$$\lim_{t\to\infty} At^{\frac{n}{2}(1-\frac{1}{q})-\frac{\sigma}{2}} \|V(\cdot,t+1) - V(\cdot,t)\|_{q,\varphi_\sigma(t)} = 0. \tag{5.5}$$

First, note that

$$V(x,t) = t^\sigma |x|^{-\sigma} G(x,t).$$

Next, using the definition of the weight $\varphi_\sigma(x,t)$ and the norm $\|\cdot\|_{q,\varphi_\sigma(x,t)}$ for $1 \leqslant q < \infty$, we obtain

$$\begin{aligned} L_q(t) &= t^{\frac{n}{2}(1-\frac{1}{q})-\frac{\sigma}{2}} \|V(\cdot,t+1) - V(\cdot,t)\|_{q,\varphi_\sigma(t)} \\ &\leqslant t^{\frac{n}{2}(1-\frac{1}{q})-\sigma+\frac{\sigma}{q}} \left(\int_{|x|\leqslant\sqrt{t}} |x|^{-2\sigma} \Big|(t+1)^\sigma G(x,t+1) - t^\sigma G(x,t)\Big|^q \mathrm{d}x\right)^{\frac{1}{q}} \\ &\quad + t^{\frac{n}{2}(1-\frac{1}{q})-\frac{\sigma}{2}} \left(\int_{|x|\geqslant\sqrt{t}} |x|^{-\sigma q} \Big|(t+1)^\sigma G(x,t+1) - t^\sigma G(x,t)\Big|^q \mathrm{d}x\right)^{\frac{1}{q}}. \end{aligned}$$

Now, substituting $x = \sqrt{t}y$, we arrive at

$$\begin{aligned} L_q(t) \;\leqslant& \left(\int_{|y|\leqslant 1} |y|^{-2\sigma} \Big|\Big(1+\frac{1}{t}\Big)^\sigma G\Big(y, 1+\frac{1}{t}\Big) - G(y,1)\Big|^q \mathrm{d}y\right)^{\frac{1}{q}} \\ &+ \left(\int_{|y|\geqslant 1} |y|^{-\sigma q} \Big|\Big(1+\frac{1}{t}\Big)^\sigma G\Big(y, 1+\frac{1}{t}\Big) - G(y,1)\Big|^q \mathrm{d}y\right)^{\frac{1}{q}}. \end{aligned}$$

Applying the Lebesgue dominated theorem, since the function $|y|^{-2\sigma}$ is locally integrable and the function $|y|^{-\sigma q}\Big|\Big(1+\frac{1}{t}\Big)^\sigma G\Big(y,1+\frac{1}{t}\Big) - G(y,1)\Big|$ is dominated by an integrable function for all $t > 1$, we get $\lim_{t\to\infty} L_q(t) = 0$.

It remains to prove (5.5) for $q = \infty$. Using the definition of the weighted norm, we get

$$\begin{aligned}L_\infty(t) &= t^{\frac{n-\sigma}{2}} \|V(\cdot, t+1) - V(\cdot, t)\|_{\infty,\varphi_\sigma(t)} \\ &\leqslant t^{\frac{n}{2}} \sup_{x \leqslant \sqrt{t}} |x|^{-2\sigma} \Big|\Big(t+1\Big)^\sigma G(x, t+1) - t^\sigma G(x, t)\Big| \\ &\quad + t^{\frac{n-\sigma}{2}} \sup_{x \geqslant \sqrt{t}} |x|^{-\sigma} \Big|\Big(t+1\Big)^\sigma G(x, t+1) - t^\sigma G(x, t)\Big|.\end{aligned}$$

Just as before, we substitute $x = \sqrt{t}y$ to obtain

$$\begin{aligned}L_\infty(t) &= \sup_{|y| \leqslant 1} \Big|\Big(1 + \frac{1}{t}\Big)^\sigma G(y, 1 + \frac{1}{t}) - G(y, 1)\Big| \\ &\quad + \sup_{|y| \geqslant 1} |y|^{-\sigma} \Big|\Big(1 + \frac{1}{t}\Big)^\sigma G(y, 1 + \frac{1}{t}) - G(y, 1)\Big|\end{aligned}$$

Since the function $|y|^{-\sigma}\Big|\Big(1 + \frac{1}{t}\Big)^\sigma G(y, 1 + \frac{1}{t}) - G(y, 1)\Big|$ is uniformly continuous for $|y| \geqslant 1$, we get $\lim_{t\to\infty} L_\infty(t) = 0$. This finishes the proof. □

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